

SIX LECTURES ABOUT (ADVANCED) STATISTICAL PHYSICS

T.S.Biró, MTA Wigner Research Centre for Physics, Budapest

Lectures given at: University of Johannesburg, South-Africa,

November 26 – November 29, 2012.

- 1. Ancient Thermodynamics (... - 1870)**
- 2. The Rise of Statistical Physics (1890 – 1920)**
- 3. Modern (postwar) Problems (1940 – 1980)**
- 4. Corrections (1950 – 2005)**
- 5. Generalizations (1960 – 2010)**
- 6. High Energy Physics (1950 – 2010)**

LECTURE ONE ABOUT (ADVANCED) STATISTICAL PHYSICS

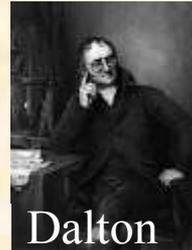
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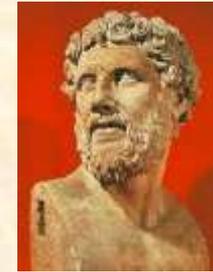
November 26, 2012.

ANCIENT THERMODYNAMICS

- **Atom**



Dalton

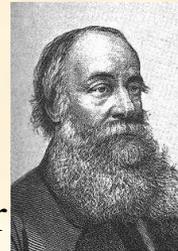


Demokritos

- **Energy**



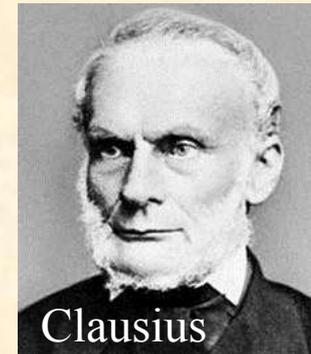
Robert-Mayer



Joule



Helmholtz



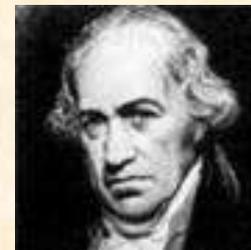
Clausius

- **Entropy**

- **Temperature**



Celsius



Fahrenheit



Kelvin



Julius Robert von Mayer, 1814 - 1878

Heat \leftrightarrow work \sim energy

Bemerkungen über die Kräfte der belebten und unbelebten Natur,
Annalen der Chemie und Pharmacie
(ed. Justus Liebig), Mai **1842**

$$1 \text{ kcal} = 425 \text{ mkp}$$

The venal blood of sailors in tropical routes is **rather blue**, in cooler zones **rather red** \rightarrow
The metabolism produces heat.

J. R. Mayer



James Prescott Joule, 1818 - 1889
Electronic heat production. 1845
Mechanical equivalent of heat. 1847

$$1 \text{ kcal} = 427 \text{ mkp}$$

$$P = R \cdot I^2$$

His student: William Thompson → Lord Kelvin





Rudolf Clausius,

Entropy: $dS = \frac{\delta Q}{T}$

The change in entropy is independent of the path!

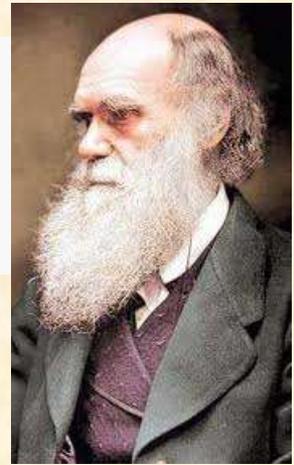
On a closed, reversible path it vanishes.

In a closed system the entropy is not decreasing spontaneously.

The heat flows from the hotter body to the cooler one.



2. Law and Life



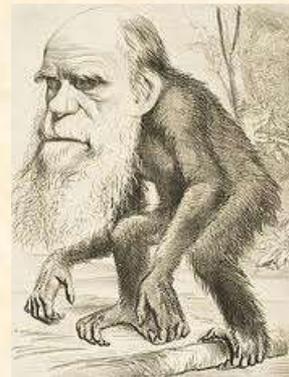
Sun: 1
Earth: 2

$$dS = \beta_1 dE_1 + \beta_2 dE_2 \geq 0$$

$$dE_1 + dE_2 = 0$$

$$dE_1 < 0, \quad (\beta_1 - \beta_2) dE_1 \geq 0$$

$$\Rightarrow \beta_1 \leq \beta_2 \quad dS_2 \geq 0$$



The entropy of Earth cannot decrease → it could be no evolution!

2. Law and Life

Sun: 1
Earth: 2
Space: 3

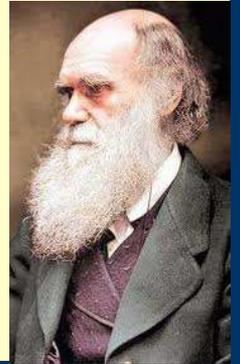
$$dS = \beta_1 dE_1 + \beta_2 dE_2 + \beta_3 dE_3 \geq 0$$

$$dE_1 + dE_2 + dE_3 = 0$$

$$dE_3 > 0, \quad dE_1 = -\lambda dE_3, \quad dE_2 = -(1 - \lambda)dE_3$$

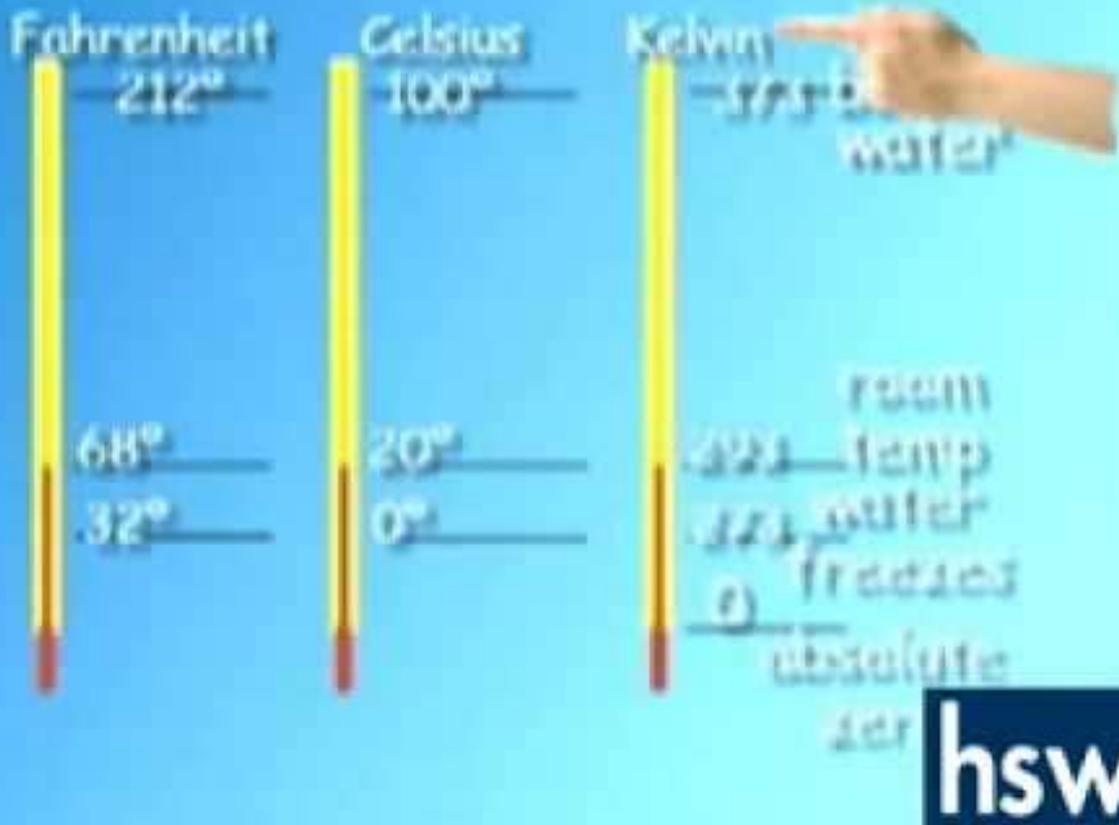
$$(\beta_3 - \beta_2) + \lambda(\beta_2 - \beta_1) \geq 0$$

$$\beta_1 < \beta_2 < \beta_3 : \quad dE_2 < 0 \quad dS_2 < 0$$



Earth' entropy can be reduced → evolution is possible!





$$\frac{F}{16} = \frac{C}{9} + 2$$



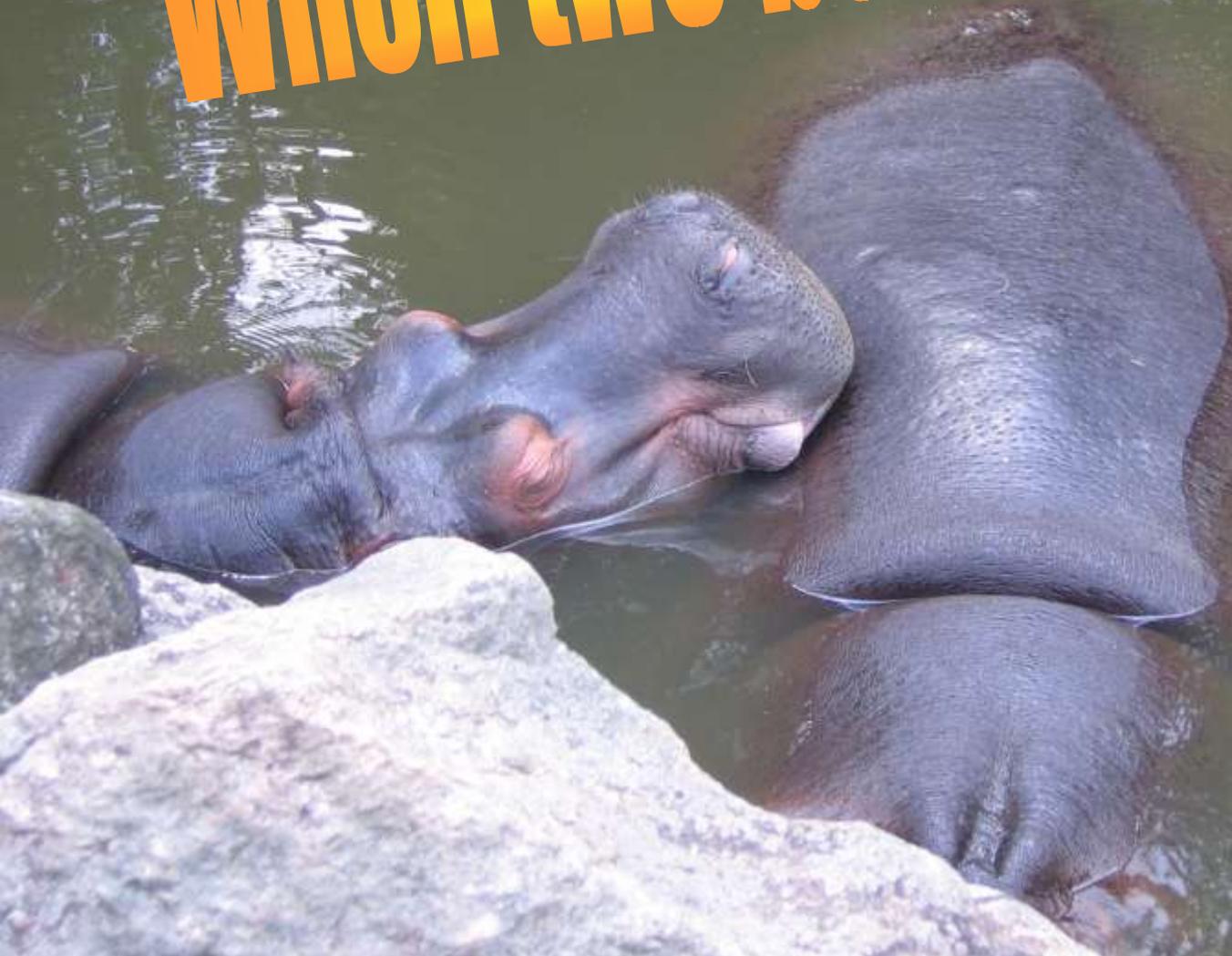


William Thomson Lord Kelvin,

Absolute temperature $T \geq 0$

„It is impossible to build a flying structure,
which is heavier than air.”

When two bodies touch....



The Zeroth Law

- Rankine 1853: Definition of equal temperature: two material bodies have equal temperature, if neither gives heat to the other.
- Maxwell 1872: When two bodies are in thermal contact, then one of them loses heat, the other gains heat, and the one which gives heat, should be considered as having the higher temperature. If none of them takes or gives heat, then they have equal temperature and are in thermal equilibrium.
- Tait 1884: If A and B, further B and C have equal temperature, then also A and C do.
- Planck 1897: If A is in thermal equilibrium with B and C, then also B and C are in thermal equilibrium.
- Clausius, Boltzmann, Jaynes: **In equilibrium entropy is maximal.**
- Fowler 1939, Fowler & Guggenheim 1965: *''Zeroth Law''*

Tamás Sándor Biró

Is There a Temperature?

Conceptual Challenges at High Energy,
Acceleration and Complexity



Jaynes' entropy maximum principle

$$S_{12}(E_{12}, V_{12}, N_{12}, \dots) = \max$$

$$E_{12} = E_1 \oplus E_2 = \text{fix}$$

$$V_{12} = V_1 \oplus V_2 = \text{fix}$$

$$N_{12} = N_1 \oplus N_2 = \text{fix}$$



The differentials are NOT independent!

Zeroth Law: $\theta(E_1, \dots) = \theta(E_2, \dots)$

Empirical temperature can be anything that equals

$$dS_{12} = \frac{\partial S_{12}}{\partial E_1} dE_1 + \frac{\partial S_{12}}{\partial E_2} dE_2 + \dots = 0$$

$$dE_{12} = \frac{\partial E_{12}}{\partial E_1} dE_1 + \frac{\partial E_{12}}{\partial E_2} dE_2 = 0$$

$$\frac{\partial E_{12}}{\partial E_2} \frac{\partial S_{12}}{\partial S_1} S'_1 = \frac{\partial E_{12}}{\partial E_1} \frac{\partial S_{12}}{\partial S_2} S'_2$$

For the addition rule this factorizes !

Entropy and energy are additive, the temperature equals

$$\frac{\partial S_{12}}{\partial S_1} \frac{\partial E_{12}}{\partial E_2} S'_1(E_1) = \frac{\partial S_{12}}{\partial S_2} \frac{\partial E_{12}}{\partial E_1} S'_2(E_2)$$

$$S_{12} = S_1 + S_2, \quad E_{12} = E_1 + E_2$$

$$\frac{1}{T_1} = S'_1(E_1) = S'_2(E_2) = \frac{1}{T_2}$$

Extensive thermodynamical equilibrium

$$S_{12} = S_1 + S_2 \quad E_{12} = E_1 + E_2$$

$$\beta = \frac{1}{T} = \frac{\partial S}{\partial E}$$

$$S[E_1, E_2] - \beta \sum_i E_i = \max$$

Extensive thermodynamical equilibrium: many states

$$S[w_i] - \beta \sum_i w_i E_i - \alpha \sum_i w_i = \max$$

w_i means the occupation ratio for the state i with energy E_i among the many examples (system-copies, ensemble-elements) . /Gibbs/

STATISTICAL PHYSICS

- **Kinetic Theory**



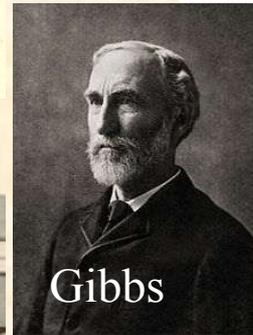
Ludwig Boltzmann

Boltzmann



Maxwell

- **Statistical Ensemble**



Gibbs



Wiener

- **Diffusion**



Brown



Einstein et.al.



Smoluchowski

- **Combinatorics**



Fermi



Bose



Planck - Fokker



Ludwig Boltzmann,



Boltzmann's entropy formula

$$S = k \log W$$

If

$$W_{12} = W_1 \cdot W_2$$

then

$$S_{12} = S_1 + S_2$$

Logarithm: product \rightarrow sum

additive
commutative
associative
extensive

$$S = \sum f \ln \frac{1}{f} \Leftrightarrow S[f_1 \cdot f_2] = S[f_1] + S[f_2]$$

$$f = \frac{1}{Z} e^{-\beta E} \Leftrightarrow f(E_1 + E_2) = f(E_1) \cdot f(E_2)$$

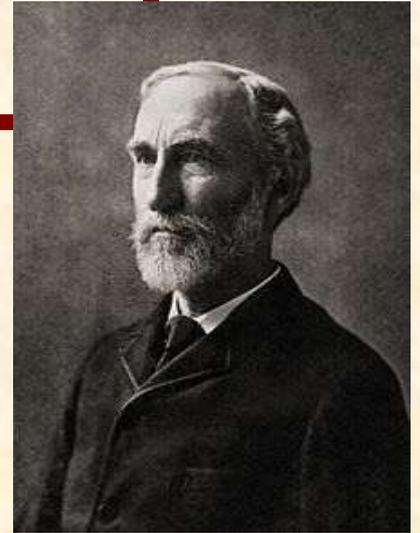
Boltzmann's entropy formula

- $S = k \log W$
- Independent permutations: $W = N!$
- Stirling formula: $\log N! \approx N \log N$
- Repeated permutations:

$$W = \frac{N!}{\prod_i N_i!}$$

- Probability: $w_i = \lim_{\{N \rightarrow \infty\}} \frac{N_i}{N}$

Gibbs' derivation



$$\ln W = N \ln N - \sum_i N_i \ln N_i$$

$$\ln W = N \ln N - \sum_i N w_i \ln(N w_i)$$

$$\ln W = N \ln N \left(1 - \sum_i w_i\right) + N \left(- \sum_i w_i \ln w_i\right)$$

$$S = -k \sum_i w_i \ln w_i \quad \text{while} \quad \sum_i w_i = 1.$$

Boltzmann-Gibbs Entropy: Extensive

$$S_{\text{Boltzmann-Gibbs}} = \sum_i w_i \ln \frac{1}{w_i}$$

$$w_i^{\text{eq}} = \frac{1}{Z} e^{-(\beta E_i + \alpha)}$$

Additive entropy



Equilibrium distribution factorizes



additive energy

$$e^{-\beta E_{12}} = e^{-\beta E_1} \cdot e^{-\beta E_2}$$

$$E_{12} = E_1 + E_2$$

Ideal Gas

N particles share a total energy E in volume V:

$$W = \int \prod_{k=1}^n d\Gamma_k \delta \left(\sum_{i=0}^n (E_i + m) - (E + M) \right)$$

$$d\Gamma_k \propto dV \cdot E_k^f dE_k$$

$$W \propto k_N V^N (E + M - Nm)^{fN-1}$$

$$S(E) = N \ln V + (fN - 1) \ln (E + M - Nm) + S_0(N)$$

Ideal Gas

Equation of state $S(E)$ and its consequences:

$$S(E) = N \ln V + (fN - 1) \ln (E + M - Nm) + S_0(N)$$

$$\frac{1}{T} = \left. \frac{\partial S}{\partial E} \right|_{V,N} = \frac{fN - 1}{E + M - Nm} \quad \text{Equipartition law}$$

$$\frac{p}{T} = \left. \frac{\partial S}{\partial V} \right|_{E,N} = \frac{N}{V} \quad \text{Boyle-Mariotte law}$$

Ideal Gas

Constant heat capacity eos:

$$\frac{1}{C_0} = -\frac{S''(E)}{S'(E)^2} = \text{const.}$$

$$\frac{1}{S'(E)} = T_0 + \frac{1}{C_0} E = T$$

$$S(E) = C_0 \ln \left(1 + \frac{E}{C_0 T_0} \right) + S_0$$

Ideal Gas

State probabilities:

$$P_{alone}(E) = K e^{-S(E)} = K \left(1 + \frac{E}{C_0 T_0} \right)^{-C_0}$$

$$P_{condition}(E_1) = \frac{P_{alone}(E)}{P_{alone}(E - E_1)} = \left(1 - \frac{E_1}{C_0 T} \right)^{C_0}$$

The law of big numbers

Distribution of the sum of n random variables:

$$P_n(x) = \int \prod_{i=1}^n dx_i w_i(x_i) \delta\left(x - a_n \sum_{k=1}^n x_k\right)$$

Fourier transform:

$$\tilde{P}_n(k) = \int dx e^{ikx} P_n(x) = \prod_{i=1}^n \tilde{w}_i(a_n k)$$

The law of big numbers

Taylor expansion of $\ln \tilde{P}_n(k) \rightarrow$ central moments

Scaling of ℓ -th moment:

$$\sigma_n^{(\ell)} = a_n^\ell n \bar{\sigma}^{(\ell)}$$

with $\bar{\sigma}$ average individual ℓ -th moment.

If all moments for $\ell < \ell_0$ are explicitly zero, and the higher moments are finite, one takes $a_n \sim n^{-1/\ell_0}$

$$\sigma_n^{(\ell)} = n^{1-\ell/\ell_0} \bar{\sigma}^{(\ell)}$$

All $\ell > \ell_0$ moments $\rightarrow 0$
for $n \rightarrow \infty$

The law of big numbers

- zeroth log moment is zero due to normalization
- first log moment can be zero due to symmetry
- second log moment is usually finite
- Consequence: for the sum of many such variables **all the higher log moments vanish**, i.e. P_n is **Gaussian!**

(central limit theorem)

Sum of uniform randoms

TSB, BM: PLB 578, 78, 2004

Let x_i be distributed uniformly in $(-1,1)$.

The distribution of

$$x = \sqrt{\frac{3}{n}} \sum_{i=1}^n x_i$$

is Gaussian.

$$\tilde{P}_n(k) = \left(\frac{\sin(k\sqrt{3/n})}{k\sqrt{3/n}} \right)^n \rightarrow \exp\left(-\frac{k^2}{2}\right)$$

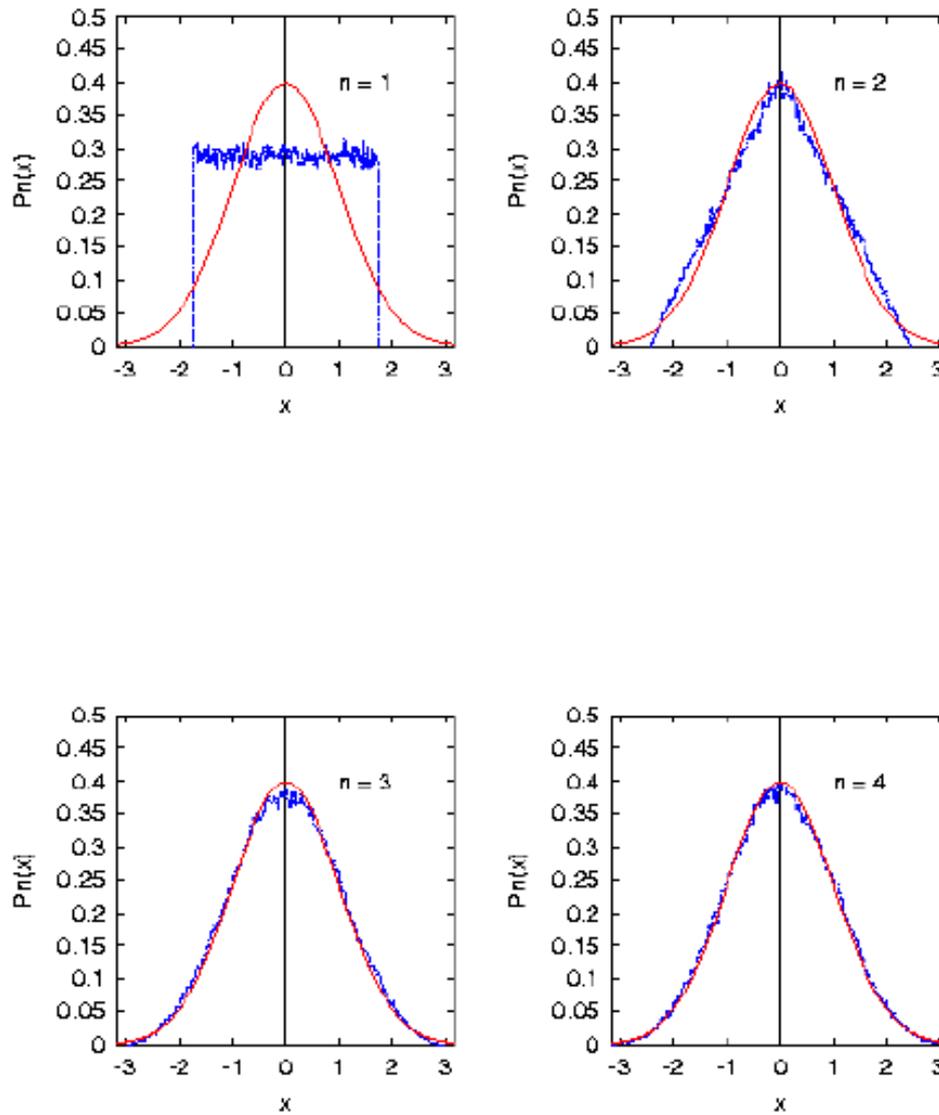


Figure 1: Comparison of histograms of $m = 200000$ sums of n uniform random deviates in $(-1, 1)$ scaled with $\sqrt{3/n}$ and the limiting Gauss distribution $\frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$.

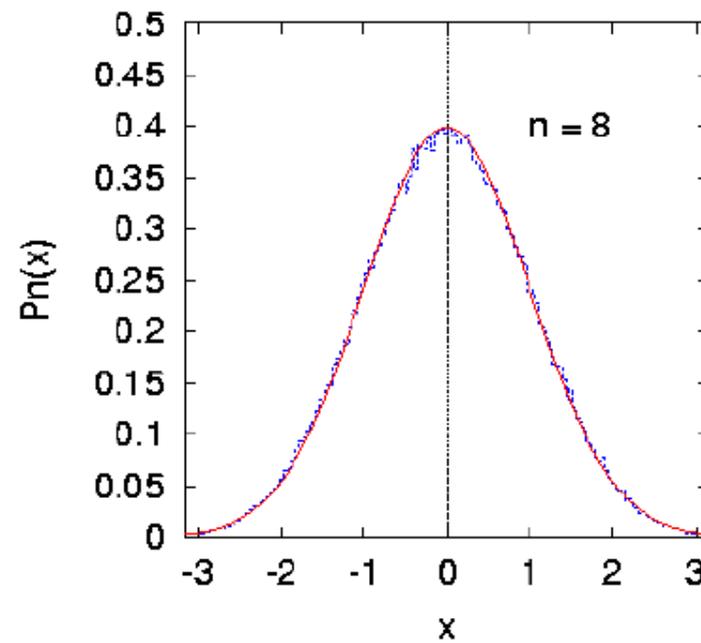


Figure 2: Comparison of histograms of $m = 200000$ sums of 8 uniform random deviates in $(-1, 1)$ scaled with $\sqrt{3/8}$ and the limiting Gauss distribution $\frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$.

Convolution of Lorentzians

Let x_i be distributed as Lorentzian.

The distribution of

$$x = \frac{1}{n} \sum_{i=1}^n x_i$$

is also Lorentzian.

$$\tilde{P}_n(k) = \left(e^{-|k|/n} \right)^n \rightarrow e^{-|k|}$$

Lévy distribution

Discussion





Is acceleration a heat container?